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University College London
DEPARTMENT OF MATHEMATICS
Mid-sessional Examinations 2005
Mathematics M11A
Wednesday 12 January 2005, 1.30-3.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to converge to a limit a as $n \to \infty$.

   (b) Prove that if $x_n \to a$ and $y_n \to b$ as $n \to \infty$ then $x_n y_n \to ab$ as $n \to \infty$.

   (c) Prove that if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.

   (d) Determine the limit as $n \to \infty$ of

$$\frac{(n+1)^{2005}}{2n^{2005}+1}.$$

2. (a) Define what it means for a to be an *upper bound* for a set S. Define what it means for a to be a *least upper bound.*

   (b) *State* the Least Upper Bound principle.

   (c) Prove that an increasing sequence $(x_n)_{n=1}^{\infty}$ that is bounded above converges to a limit.

   (d) State and prove the Bolzano-Weierstrass Theorem.

3. (a) Define what it means for a series $\sum_{n=1}^{\infty} x_n$ to be convergent.

   (b) *State* the Comparison Test for series.

   (c) State and prove the Ratio Test for series.

   (d) For each of the following series, determine whether or not it converges:

$$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}, \qquad \sum_{n=1}^{\infty} \frac{(3n)!}{26^n (n!)^3}, \qquad \sum_{n=1}^{\infty} \frac{n!(2n)!\sin(n^{2005})}{(3n)!}.$$

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4. (a) Define what it means to say for a function f that $f(x) \to L$ as $x \to c$.

(b) Define what it means to say that f is *continuous* at c.

(c) Prove that the following two statements are equivalent:

1. f is continuous at c
2. For every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \neq c$ and $x_n \to c$ as $n \to \infty$, we have $f(x_n) \to f(c)$ as $n \to \infty$.

(d) Suppose that g is continuous at c and f is continuous at $g(c)$. Prove that $f \circ g$ is continuous at c.

(e) Suppose that g is continuous at c and $f \circ g$ is continuous at c. Must f be continuous at $g(c)$?

5. (a) Define $\exp(x)$ for a real number x by giving a power series.

(b) *State* the Intermediate Value Theorem.

(c) Prove that for every $y > 0$ there exists a unique real number $\log y$ such that $\exp(\log y) = y$. (You may assume that $\exp(x)$ is continuous.)

(d) Prove that, for $x, y > 0$,

$$\log(xy) = \log(x) + \log(y).$$

(e) Prove that $\log(x)$ is continuous for $x > 0$.

6. (a) Define what it means for a function f to be convex on an interval I.

(b) State and prove Jensen's Inequality for convex functions.

(c) State and prove the Arithmetic Mean/Geometric Mean inequality. [You may assume that $-\log x$ is convex.]

(d) Prove that, for $n \geqslant 1$,

$$(n!)^{1/n} \leqslant \frac{n+1}{2}.$$

END OF PAPER